

The Number of Complete Maps on Surfaces

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Abstract: A map is a connected topological graph cellularly embedded in a surface and a complete map is a cellularly embedded complete graph in a surface. In this paper, all automorphisms of complete maps of order n are determined by permutations on its vertices. Applying a scheme for enumerating maps on surfaces with a given underlying graph, the numbers of unrooted complete maps on orientable or non-orientable surfaces are obtained.

Key words: embedding, complete map, isomorphism, automorphism group, Burnside Lemma.

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1. Introduction

All surfaces considered in this paper are 2-dimensional compact closed manifolds without boundary, graphs are connected and simple graphs with the maximum valency ≥ 3 and groups are finite. For terminologies and notations not defined here can be seen in [21] for maps, [20] for graphs and in [2] for permutation groups.

The enumeration of rooted maps on surfaces, especially, the sphere, has been intensively investigated by many researchers after the Tutte's pioneer work in 1962 (see [21]). Comparing with rooted maps, observation for the enumeration of unrooted maps on surface is not much. By applying the automorphisms of the sphere, Liskovets gave an enumerative scheme for unrooted planar maps(see [12]). Liskovets, Walsh and Liskovets got many enumeration results for *general planar maps*, *regular planar maps*, *Eulerian planar maps*, *self-dual planar maps* and *2-connected planar maps*, etc (see [12] – [14]).

General results for the enumeration of unrooted maps on surface other than sphere are very few. Using the well known Burnside Lemma in permutation group theory, Biggs and White presented a formula for enumerating non-equivalent em-

beddings of a given graph on orientable surfaces^[2], which are the classification of embeddings by orientation-preserving automorphisms of orientable surfaces. Following their idea, the numbers of non-equivalent embeddings of complete graphs, complete bipartite graphs, wheels and graphs whose automorphism group action on its ordered pair of adjacent vertices is semi-regular are gotten in references [15] – [16], [20] and [11]. Although this formula is not very efficient and need more clarifying for the actual enumeration of non-equivalent embeddings of a graph, the same idea is more practical for enumerating rooted maps on orientable or non-orientable surfaces with given underlying graphs(see [8] – [10]).

For projective maps with a given 3-connected underlying graph, Negami got an enumeration result for non-equivalent embeddings by establishing the double planar covering of projective maps(see [18]). In [7], Jin Ho Kwak and Jaeun Lee obtained the number of non-congruent embeddings of a graph, which is also related to the topic discussed in this paper.

Combining the idea of Biggs and White for non-equivalent embeddings of a graph on orientable surfaces and the Tutte's algebraic representation for maps on surface^{[19],[21]}, a general scheme for enumerating unrooted maps on locally orientable surfaces with a given underlying graph is obtained in this paper. Whence, the enumeration of unrooted maps on surfaces can be carried out by the following programming:

- STEP 1. Determined all automorphisms of maps with a given underlying graph;
- STEP 2. Calculation the the fixing set $Fix(\varsigma)$ for each automorphism ς of maps;
- STEP 3. Enumerating the unrooted maps on surfaces with a given underlying graph by this scheme.

Notice that this programming can be used for orientable or non-orientable surfaces, respectively and get the numbers of orientable or non-orientable unrooted maps underlying a given graph.

The main purpose of this paper is to enumerate the orientable or non-orientable complete maps. In 1971, Biggs proved^[1] that the order of automorphism group of an orientable complete map of order n divides $n(n - 1)$, and equal $n(n - 1)$ only if the automorphism group of the complete map is a *Frobenius group*. In this paper, we get a representation by the permutation on its vertices for the automorphisms of orientable or non-orientable complete maps. Then as soon as we completely calculate the fixing set $Fix(\varsigma)$ for each automorphism ς of complete maps, the enumeration of unrooted orientable or non-orientable complete maps can be well done by our programming.

The problem of determining which automorphism of a graph is an automorphism of a map is also interesting for *Riemann* surfaces or *Klein* surfaces - surfaces equipped with an analytic or dianalytic structure, for example, automorphisms of Riemann or Klein surfaces have be given more attention since 1960s, see for example,[3] – [4], [6], [17], but it is difficult to get a concrete representation for an automorphism

of Riemann or Klein surfaces. The approach used in this paper can be also used for combinatorial discussion automorphisms of Riemann or Klein surface.

Terminologies and notations used in this paper are standard. Some of them are mentioned in the following.

For a given connected graph Γ , an *embedding* of Γ is a pair (\mathcal{J}, λ) , where \mathcal{J} is a rotation system of Γ , and $\lambda : E(\Gamma) \rightarrow \mathbb{Z}_2$. The edge with $\lambda(e) = 0$ or $\lambda(e) = 1$ is called the *type 0* or *type 1 edge*, respectively.

A *map* $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is defined to be a permutation \mathcal{P} acting on $\mathcal{X}_{\alpha,\beta}$ of a disjoint union of quadricells Kx of $x \in X$, where, X is a finite set and $K = \{1, \alpha, \beta, \alpha\beta\}$ is the *Klein* group, satisfying the following conditions:

- i) $\forall x \in \mathcal{X}_{\alpha,\beta}$, there does not exist an integer k such that $\mathcal{P}^k x = \alpha x$;
- ii) $\alpha \mathcal{P} = \mathcal{P}^{-1} \alpha$;
- iii) the group $\Psi_J = \langle \alpha, \beta, \mathcal{P} \rangle$ is transitive on $\mathcal{X}_{\alpha,\beta}$.

According to the condition *ii*), the vertices of a map are defined to be the pairs of conjugate of \mathcal{P} action on $\mathcal{X}_{\alpha,\beta}$ and edges the orbits of K on $\mathcal{X}_{\alpha,\beta}$, for example, for $\forall x \in \mathcal{X}_{\alpha,\beta}$, $\{x, \alpha x, \beta x, \alpha\beta x\}$ is an edge of the map M . Geometrically, any map M is an embedding of a graph Γ on a surface, denoted by $M = M(\Gamma)$ and $\Gamma = \Gamma(M)$ (see also [19] – [21] for details). The graph Γ is called the underlying graph of the map M . If $r \in \mathcal{X}_{\alpha,\beta}$ is marked beforehand, then M is called a *rooted map*, denoted by M^r . A map is said non-orientable or orientable if the group $\Psi_I = \langle \alpha\beta, \mathcal{P} \rangle$ is transitive on $\mathcal{X}_{\alpha,\beta}$ or not.

For example, the graph K_4 on the tours with one face length 4 and another 8 , shown in the following Fig.1,

Fig.1

can be algebraically represented as follows:

A map $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ with $\mathcal{X}_{\alpha,\beta} = \{x, y, z, u, v, w, \alpha x, \alpha y, \alpha z, \alpha u, \alpha v, \alpha w, \beta x, \beta y, \beta z, \beta u, \beta v, \beta w, \alpha\beta x, \alpha\beta y, \alpha\beta z, \alpha\beta u, \alpha\beta v, \alpha\beta w\}$ and

$$\begin{aligned} \mathcal{P} &= (x, y, z)(\alpha\beta x, u, w)(\alpha\beta z, \alpha\beta u, v)(\alpha\beta y, \alpha\beta v, \alpha\beta w) \\ &\times (\alpha x, \alpha z, \alpha y)(\beta x, \alpha w, \alpha u)(\beta z, \alpha v, \beta u)(\beta y, \beta w, \beta v) \end{aligned}$$

The four vertices of this map are $\{(x, y, z), (\alpha x, \alpha z, \alpha y)\}$, $\{(\alpha\beta x, u, w), (\beta x, \alpha w, \alpha u)\}$, $\{(\alpha\beta z, \alpha\beta u, v), (\beta z, \alpha v, \beta u)\}$ and $\{(\alpha\beta y, \alpha\beta v, \alpha\beta w), (\beta y, \beta w, \beta v)\}$ and six edges are $\{e, \alpha e, \beta e, \alpha\beta e\}$ for $\forall e \in \{x, y, z, u, v, w\}$.

Two maps $M_1 = (\mathcal{X}_{\alpha, \beta}^1, \mathcal{P}_1)$ and $M_2 = (\mathcal{X}_{\alpha, \beta}^2, \mathcal{P}_2)$ are said to be *isomorphic* if there exists a bijection $\tau : \mathcal{X}_{\alpha, \beta}^1 \longrightarrow \mathcal{X}_{\alpha, \beta}^2$ such that for $\forall x \in \mathcal{X}_{\alpha, \beta}^1, \tau\alpha(x) = \alpha\tau(x)$, $\tau\beta(x) = \beta\tau(x)$ and $\tau\mathcal{P}_1(x) = \mathcal{P}_2\tau(x)$. τ is called an *isomorphism* between them. If $M_1 = M_2 = M$, then an isomorphism between M_1 and M_2 is called an *automorphism* of M . All automorphisms of a map M form a group, called the *automorphism group* of M and denoted by $\text{Aut}M$. Similarly, two rooted maps M_1^r, M_2^r are said to be *isomorphic* if there is an isomorphism θ between them such that $\theta(r_1) = r_2$, where r_1, r_2 are the roots of M_1^r, M_2^r , respectively and denote the *automorphism group* of M^r by $\text{Aut}M^r$. It has been known that $\text{Aut}M^r$ is a trivial group.

According to their action, isomorphisms between maps can be divided into two classes: *cyclic order-preserving isomorphism* and *cyclic order-reversing isomorphism*, defined as follows, which is useful for determining automorphisms of a map underlying a graph.

For two maps M_1 and M_2 , a bijection ξ between M_1 and M_2 is said to be *cyclic order-preserving* if for $\forall x \in \mathcal{X}_{\alpha, \beta}^1, \tau\alpha(x) = \alpha\tau(x)$, $\tau\beta(x) = \beta\tau(x)$, $\tau\mathcal{P}_1(x) = \mathcal{P}_2\tau(x)$ and *cyclic order-reversing* if $\tau\alpha(x) = \alpha\tau(x)$, $\tau\beta(x) = \beta\tau(x)$, $\tau\mathcal{P}_1(x) = \mathcal{P}_2^{-1}\tau(x)$.

Now let Γ be a connected graph. The notations $\mathcal{E}^O(\Gamma)$, $\mathcal{E}^N(\Gamma)$ and $\mathcal{E}^L(\Gamma)$ denote the embeddings of Γ on the orientable surfaces, non-orientable surfaces and locally surfaces, $\mathcal{M}(\Gamma)$ and $\text{Aut}\Gamma$ denote the set of non-isomorphic maps underlying Γ and its automorphism group, respectively.

2. The enumerative scheme for maps underlying a graph

A permutation p on set Ω is called semi-regular if all of its orbits have the same length. For a given connected graph Γ , $\forall g \in \text{Aut}\Gamma$, $M = (\mathcal{X}_{\alpha, \beta}, \mathcal{P}) \in \mathcal{M}(\Gamma)$, define an extended action of g on M to be

$$g^* : \mathcal{X}_{\alpha, \beta} \longrightarrow \mathcal{X}_{\alpha, \beta},$$

such that $M^{g^*} = gMg^{-1}$ with $g\alpha = \alpha g$ and $g\beta = \beta g$.

We have already known the following two results.

Lemma 2.1^[21] *For any rooted map M^r , $\text{Aut}M^r$ is trivial.*

Lemma 2.2^{[2],[21]} *For a given map M , $\forall \xi \in \text{Aut}M$, ξ transforms vertices to vertices,*

edges to edges and faces to faces on a map M , i.e, ξ can be naturally extended to an automorphism of surfaces.

Lemma 2.3 *If there is an isomorphism ξ between maps M_1 and M_2 , then $\Gamma(M_1) = \Gamma(M_2) = \Gamma$ and $\xi \in \text{Aut}\Gamma$ if ξ is cyclic order-preserving or $\xi\alpha \in \text{Aut}\Gamma$ if ξ is cyclic order-reversing.*

Proof By the definition of an isomorphism between maps, if $M_1 = (\mathcal{X}_{\alpha,\beta}^1, \mathcal{P}_1)$ is isomorphic with $M_2 = (\mathcal{X}_{\alpha,\beta}^2, \mathcal{P}_2)$, then there is an 1 – 1 mapping ξ between $\mathcal{X}_{\alpha,\beta}^1$ and $\mathcal{X}_{\alpha,\beta}^2$ such that $(\mathcal{P}_1)^\xi = \mathcal{P}_2$. Since isomorphic graphs are considered to be equal, we get that $\Gamma(M_1) = \Gamma(M_2) = \Gamma$. Now since

$$(\mathcal{P}_2)^{-1} = (\mathcal{P}_2)^\alpha.$$

We get that $\Gamma^\xi = \Gamma$ or $\Gamma^{\xi\alpha} = \Gamma$, whence, $\xi \in \text{Aut}\Gamma$ or $\xi\alpha \in \text{Aut}\Gamma$. \spadesuit

According to Lemma 2.3, For $\forall g \in \text{Aut}\Gamma, \forall M \in \mathcal{E}^L(\Gamma)$, the induced action g^* of g on M is defined by $M^{g^*} = gMg^{-1} = (\mathcal{X}_{\alpha,\beta}, g\mathcal{P}g^{-1})$.

Since \mathcal{P} is a permutation on the set $\mathcal{X}_{\alpha,\beta}$, by a simple result in permutation group theory, \mathcal{P}^g is just the permutation replaced each element x in \mathcal{P} by $g(x)$. Whence M and M^{g^*} are isomorphic. Therefore, we get the following enumerative theorem for unrooted maps underlying a graph.

Theorem 2.1 *For a connected graph Γ , let $\mathcal{E} \subset \mathcal{E}^L(\Gamma)$. Then the number $n(\mathcal{E}, \Gamma)$ of unrooted maps in \mathcal{E} is*

$$n(\mathcal{E}, \Gamma) = \frac{1}{|\text{Aut}\Gamma \times \langle \alpha \rangle|} \sum_{g \in \text{Aut}\Gamma \times \langle \alpha \rangle} |\Phi(g)|,$$

where, $\Phi(g) = \{\mathcal{P} | \mathcal{P} \in \mathcal{E} \text{ and } \mathcal{P}^g = \mathcal{P}\}$.

Proof According to Lemma 2.1, two maps $M_1, M_2 \in \mathcal{E}$ are isomorphic if and only if there exists an isomorphism $\theta \in \text{Aut}\Gamma \times \langle \alpha \rangle$ such that $M_1^{\theta^*} = M_2$. Whence, we get that all the unrooted maps in \mathcal{E} are just the representations of orbits in \mathcal{E} under the action of $\text{Aut}\Gamma \times \langle \alpha \rangle$. By the Burnside Lemma, we get the following result for the number of unrooted maps in \mathcal{E}

$$n(\mathcal{E}, \Gamma) = \frac{1}{|\text{Aut}\Gamma \times \langle \alpha \rangle|} \sum_{g \in \text{Aut}\Gamma \times \langle \alpha \rangle} |\Phi(g)|. \quad \spadesuit$$

Corollary 2.1 *For a given graph Γ , the numbers of unrooted maps in $\mathcal{E}^O(\Gamma), \mathcal{E}^N(\Gamma)$ and $\mathcal{E}^L(\Gamma)$ are*

$$n^O(\Gamma) = \frac{1}{|\text{Aut}\Gamma \times \langle \alpha \rangle|} \sum_{g \in \text{Aut}\Gamma \times \langle \alpha \rangle} |\Phi^O(g)|; \quad (2.1)$$

$$n^N(\Gamma) = \frac{1}{|\text{Aut}\Gamma \times \langle \alpha \rangle|} \sum_{g \in \text{Aut}\Gamma \times \langle \alpha \rangle} |\Phi^N(g)|; \quad (2.2)$$

$$n^L(\Gamma) = \frac{1}{|\text{Aut}\Gamma \times \langle \alpha \rangle|} \sum_{g \in \text{Aut}\Gamma \times \langle \alpha \rangle} |\Phi^L(g)|, \quad (2.3)$$

where, $\Phi^O(g) = \{\mathcal{P} | \mathcal{P} \in \mathcal{E}^O(\Gamma) \text{ and } \mathcal{P}^g = \mathcal{P}\}$, $\Phi^N(g) = \{\mathcal{P} | \mathcal{P} \in \mathcal{E}^N(\Gamma) \text{ and } \mathcal{P}^g = \mathcal{P}\}$, $\Phi^L(g) = \{\mathcal{P} | \mathcal{P} \in \mathcal{E}^L(\Gamma) \text{ and } \mathcal{P}^g = \mathcal{P}\}$.

Corollary 2.2 *Let $\mathcal{E}(S, \Gamma)$ be the embeddings of Γ in the surface S , then the number $n(\Gamma, S)$ of unrooted maps on S with underlying $g\Gamma$ is*

$$n(\Gamma, S) = \frac{1}{|\text{Aut}\Gamma \times \langle \alpha \rangle|} \sum_{g \in \text{Aut}\Gamma \times \langle \alpha \rangle} |\Phi(g)|,$$

where, $\Phi(g) = \{\mathcal{P} | \mathcal{P} \in \mathcal{E}(S, \Gamma) \text{ and } \mathcal{P}^g = \mathcal{P}\}$.

Corollary 2.3 *In formulae (2.1)-(2.3), $|\Phi(g)| \neq 0$ if and only if g is an automorphism of an orientable or non-orientable map underlying Γ .*

Directly using these formulae (2.1)-(2.3) to count unrooted maps with a given underlying graph is not straightforward. More observation should be considered. The following two lemmas give necessary conditions for an induced automorphism of a graph Γ to be an cyclic order-preserving automorphism of a surface.

Lemma 2.4 *For a map M underlying a graph Γ , $\forall g \in \text{Aut}M, \forall x \in \mathcal{X}_{\alpha, \beta}$ with $X = E(\Gamma)$,*

$$(i) \quad |x^{\text{Aut}M}| = |\text{Aut}M|;$$

$$(ii) \quad |x^{<g>}| = o(g),$$

where, $o(g)$ denotes the order of g .

Proof For a subgroup $H < \text{Aut}M$, we know that $|H| = |x^H| |H_x|$. Since $H_x < \text{Aut}M^x$, where M^x is a rooted map with root x , we know that $|H_x| = 1$ by Lemma 2.1. Whence, $|x^H| = |H|$. Now take $H = \text{Aut}M$ or $<g>$, we get the assertions (i) and (ii). \square

Lemma 2.5 *Let Γ be a connected graph and $g \in \text{Aut}\Gamma$. If there is a map $M \in \mathcal{E}^L(\Gamma)$ such that the induced action $g^* \in \text{Aut}M$, then for $\forall (u, v), (x, y) \in E(\Gamma)$,*

$$[l^g(u), l^g(v)] = [l^g(x), l^g(y)] = \text{constant},$$

where, $l^g(w)$ denotes the length of the cycle containing the vertex w in the cycle decomposition of g and $[a, b]$ the least common multiple of integers a and b .

Proof According to Lemma 2.4, we know that the length of any quadricell u^{v+} or u^{v-} under the action of g^* is $[l^g(u), l^g(v)]$. Since g^* is an automorphism of map,

therefore, g^* is semi-regular. Whence, we get that

$$[l^g(u), l^g(v)] = [l^g(x), l^g(y)] = \text{constant}. \quad \spadesuit$$

Now we consider conditions for an induced automorphism of a map by an automorphism of graph to be a cyclic order-reversing automorphism of surfaces.

Lemma 2.6 *If $\xi\alpha$ is an automorphism of a map, then $\xi\alpha = \alpha\xi$.*

Proof Since $\xi\alpha$ is an automorphism of a map, we know that

$$(\xi\alpha)\alpha = \alpha(\xi\alpha).$$

That is, $\xi\alpha = \alpha\xi$. \spadesuit

Lemma 2.7 *If ξ is an automorphism of $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, then $\xi\alpha$ is semi-regular on $\mathcal{X}_{\alpha,\beta}$ with order $o(\xi)$ if $o(\xi) \equiv 0 \pmod{2}$ or $2o(\xi)$ if $o(\xi) \equiv 1 \pmod{2}$.*

Proof Since ξ is an automorphism of map by Lemma 2.6, we know that the cycle decomposition of ξ can be represented by

$$\xi = \prod_k (x_1, x_2, \dots, x_k)(\alpha x_1, \alpha x_2, \dots, \alpha x_k),$$

where, \prod_k denotes the product of disjoint cycles with length $k = o(\xi)$.

Therefore, if $k \equiv 0 \pmod{2}$, we get that

$$\xi\alpha = \prod_k (x_1, \alpha x_2, x_3, \dots, \alpha x_k)$$

and if $k \equiv 1 \pmod{2}$, we get that

$$\xi\alpha = \prod_{2k} (x_1, \alpha x_2, x_3, \dots, x_k, \alpha x_1, x_2, \alpha x_3, \dots, \alpha x_k).$$

Whence, ξ is semi-regular acting on $\mathcal{X}_{\alpha,\beta}$. \spadesuit

Now we can prove the following result for cyclic order-reversing automorphisms of maps.

Lemma 2.8 *For a connected graph Γ , let \mathcal{K} be all automorphisms in $\text{Aut}\Gamma$ whose extending action on $\mathcal{X}_{\alpha,\beta}$, $X = E(\Gamma)$, are automorphisms of maps underlying the graph Γ . Then for $\forall \xi \in \mathcal{K}$, $o(\xi^*) \geq 2$, $\xi^*\alpha \in \mathcal{K}$ if and only if $o(\xi^*) \equiv 0 \pmod{2}$.*

Proof Notice that by Lemma 2.7, if ξ^* is an automorphism of a map underlying Γ , then $\xi^*\alpha$ is semi-regular acting on $\mathcal{X}_{\alpha,\beta}$.

Assume ξ^* is an automorphism of the map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$. Without loss of generality, we assume that

$$\mathcal{P} = C_1 C_2 \cdots C_k,$$

where, $C_i = (x_{i1}, x_{i2}, \dots, x_{ij_i})$ is a cycle in the decomposition of $\xi|_{V(\Gamma)}$ and $x_{it} = \{(e^{i1}, e^{i2}, \dots, e^{iti})(\alpha e^{i1}, \alpha e^{it_i}, \dots, \alpha e^{i2})\}$,

$$\xi|_{E(\Gamma)} = (e_{11}, e_{12}, \dots, e_{s_1})(e_{21}, e_{22}, \dots, e_{2s_2}) \cdots (e_{l1}, e_{l2}, \dots, e_{ls_l}),$$

and

$$\xi^* = C(\alpha C^{-1} \alpha),$$

where, $C = (e_{11}, e_{12}, \dots, e_{s_1})(e_{21}, e_{22}, \dots, e_{2s_2}) \cdots (e_{l1}, e_{l2}, \dots, e_{ls_l})$. Now since ξ^* is an automorphism of a map, we get that $s_1 = s_2 = \dots = s_l = o(\xi^*) = s$.

If $o(\xi^*) \equiv 0(mod 2)$, define a map $M^* = (\mathcal{X}_{\alpha, \beta}, \mathcal{P}^*)$ with

$$\mathcal{P}^* = C_1^* C_2^* \cdots C_k^*,$$

where, $C_i^* = (x_{i1}^*, x_{i2}^*, \dots, x_{ij_i}^*)$, $x_{it}^* = \{(e_{i1}^*, e_{i2}^*, \dots, e_{iti}^*)(\alpha e_{i1}^*, \alpha e_{it_i}^*, \dots, e_{i2}^*)\}$ and $e_{ij}^* = e_{pq}$. Take $e_{ij}^* = e_{pq}$ if $q \equiv 1(mod 2)$ and $e_{ij}^* = \alpha e_{pq}$ if $q \equiv 0(mod 2)$. Then we get that $M^{\xi^*} = M$.

Now if $o(\xi^*) \equiv 1(mod 2)$, by Lemma 2.7, $o(\xi^* \alpha) = 2o(\xi^*)$. Therefore, for a chosen quadricell in $(e^{i1}, e^{i2}, \dots, e^{iti})$ adjacent to the vertex x_{i1} for $i = 1, 2, \dots, n$, where, n = the order of the graph Γ , the resultant map M is unstable under the action of $\xi \alpha$. Whence, $\xi \alpha$ is not an automorphism of a map underlying Γ . \spadesuit

3. Determine automorphisms of complete maps

Now we determine all automorphisms of complete maps in this section by applying the results gotten in Section 2.

Let K_n be a complete graph of order n . Label its vertices by integers $1, 2, \dots, n$. Then its edge set is $\{ij | 1 \leq i, j \leq n, i \neq j \text{ and } ij = ji\}$. For convenience, we use i^j denoting an edge ij of the complete graph K_n and $i^j = j^i, 1 \leq i, j \leq n, i \neq j$. Then its quadricells of this edge can be represented by $\{i^{j+}, i^{j-}, j^{i+}, j^{i-}\}$ and

$$\mathcal{X}_{\alpha, \beta}(K_n) = \{i^{j+} : 1 \leq i, j \leq n, i \neq j\} \cup \{i^{j-} : 1 \leq i, j \leq n, i \neq j\},$$

$$\alpha = \prod_{1 \leq i, j \leq n, i \neq j} (i^{j+}, i^{j-}),$$

$$\beta = \prod_{1 \leq i, j \leq n, i \neq j} (i^{j+}, i^{j+})(i^{j-}, i^{j-}).$$

Recall that the automorphism group of K_n is just the symmetry group of degree n , i.e., $\text{Aut} K_n = S_n$. The above representation enables us to determine all automorphisms of complete maps of order n on surfaces.

Theorem 3.1 *All cyclic order-preserving automorphisms of non-orientable complete maps of order ≥ 4 are extended actions of elements in*

$$\mathcal{E}_{[s^{\frac{n}{s}}]}, \quad \mathcal{E}_{[1, s^{\frac{n-1}{s}}]},$$

and all cyclic order-reversing automorphisms of non-orientable complete maps of order ≥ 4 are extended actions of elements in

$$\alpha\mathcal{E}_{[(2s)^{\frac{n}{2s}}]}, \quad \alpha\mathcal{E}_{[(2s)^{\frac{4}{2s}}]}, \quad \alpha\mathcal{E}_{[1,1,2]},$$

where, \mathcal{E}_θ denotes the conjugate class containing element θ in the symmetry group S_n

Proof Firstly, we prove that the induced permutation ξ^* on complete map of order n by an element $\xi \in S_n$ is an cyclic order-preserving automorphism of a non-orientable map, if, and only if,

$$\xi \in \mathcal{E}_{[s^{\frac{n}{s}}]} \cup \mathcal{E}_{[1, s^{\frac{n-1}{s}}]}$$

Assume the cycle index of ξ is $[1^{k_1}, 2^{k_2}, \dots, n^{k_n}]$. If there exist two integers $k_i, k_j \neq 0$, and $i, j \geq 2, i \neq j$, then in the cycle decomposition of ξ , there are two cycles

$$(u_1, u_2, \dots, u_i) \quad \text{and} \quad (v_1, v_2, \dots, v_j).$$

Since

$$[l^\xi(u_1), l^\xi(u_2)] = i \quad \text{and} \quad [l^\xi(v_1), l^\xi(v_2)] = j$$

and $i \neq j$, we know that ξ^* is not an automorphism of embedding by Lemma 2.5. Whence, the cycle index of ξ must be the form of $[1^k, s^l]$.

Now if $k \geq 2$, let $(u), (v)$ be two cycles of length 1 in the cycle decomposition of ξ . By Lemma 2.5, we know that

$$[l^\xi(u), l^\xi(v)] = 1.$$

If there is a cycle (w, \dots) in the cycle decomposition of ξ whose length greater or equal to two, we get that

$$[l^\xi(u), l^\xi(w)] = [1, l^\xi(w)] = l^\xi(w).$$

According to Lemma 2.5, we get that $l^\xi(w) = 1$, a contradiction. Therefore, the cycle index of ξ must be the forms of $[s^l]$ or $[1, s^l]$. Whence, $sl = n$ or $sl + 1 = n$. Calculation shows that $l = \frac{n}{s}$ or $l = \frac{n-1}{s}$. That is, the cycle index of ξ is one of the following three types $[1^n]$, $[1, s^{\frac{n-1}{s}}]$ and $[s^{\frac{n}{s}}]$ for some integer s .

Now we only need to prove that for each element ξ in $\mathcal{E}_{[1, s^{\frac{n-1}{s}}]}$ and $\mathcal{E}_{[s^{\frac{n}{s}}]}$, there exists an non-orientable complete map M of order n with an induced permutation ξ^* being its cyclic order-preserving automorphism of surface. The discussion are divided into two cases.

Case 1

$$\xi \in \mathcal{E}_{[s^{\frac{n}{s}}]}$$

Assume the cycle decomposition of ξ being $\xi = (a, b, \dots, c) \cdots (x, y, \dots, z) \cdots (u, v, \dots, w)$, where, the length of each cycle is k , and $1 \leq a, b, \dots, c, x, y, \dots, z, u, v, \dots, w \leq n$. In this case, we can construct a non-orientable complete map $M_1 = (\mathcal{X}_{\alpha, \beta}^1, \mathcal{P}_1)$ as follows.

$$\mathcal{X}_{\alpha, \beta}^1 = \{i^{j+} : 1 \leq i, j \leq n, i(j)\} \cup \{i^{j-} : 1 \leq i, j \leq n, i \neq j\},$$

$$\mathcal{P}_1 = \prod_{x \in \{a, b, \dots, c, \dots, x, y, \dots, z, u, v, \dots, w\}} (C(x))(\alpha C(x)^{-1} \alpha),$$

where,

$$C(x) = (x^{a+}, \dots, x^{x*}, \dots, x^{u+}, x^{b+}, x^{y+}, \dots, \dots, x^{v+}, x^{c+}, \dots, x^{z+}, \dots, x^{w+}),$$

x^{x*} denotes an empty position and

$$\alpha C(x)^{-1} \alpha = (x^{a-}, x^{w-}, \dots, x^{z-}, \dots, x^{c-}, x^{v-}, \dots, x^{b-}, x^{u-}, \dots, x^{y-}, \dots).$$

It is clear that $M_1^{\xi^*} = M_1$. Therefore, ξ^* is an cyclic order-preserving automorphism of the map M_1 .

Case 2 $\xi \in \mathcal{E}_{[1, s \frac{n-1}{s}]}$

We assume the cycle decomposition of ξ being

$$\xi = (a, b, \dots, c) \cdots (x, y, \dots, z) \cdots (u, v, \dots, w)(t),$$

where, the length of each cycle is k beside the final cycle, and $1 \leq a, b, \dots, c, x, y, \dots, z, u, v, \dots, w, t \leq n$. In this case, we construct a non-orientable complete map $M_2 = (\mathcal{X}_{\alpha, \beta}^2, \mathcal{P}_2)$ as follows.

$$\mathcal{X}_{\alpha, \beta}^2 = \{i^{j+} : 1 \leq i, j \leq n, i \neq j\} \cup \{i^{j-} : 1 \leq i, j \leq n, i \neq j\},$$

$$\mathcal{P}_2 = (A)(\alpha A^{-1}) \prod_{x \in \{a, b, \dots, c, \dots, x, y, \dots, z, u, v, \dots, w\}} (C(x))(\alpha C(x)^{-1} \alpha),$$

where,

$$A = (t^{a+}, t^{x+}, \dots, t^{u+}, t^{b+}, t^{y+}, \dots, t^{v+}, \dots, t^{c+}, t^{z+}, \dots, t^{w+}),$$

$$\alpha A^{-1} \alpha = (t^{a-}, t^{w-}, \dots, t^{z-}, t^{c-}, t^{v-}, \dots, t^{y-}, \dots, t^{b-}, t^{u-}, \dots, t^{x-}),$$

$$C(x) = (x^{a+}, \dots, x^{x*}, \dots, x^{u+}, x^{b+}, \dots, x^{y+}, \dots, x^{v+}, \dots, x^{c+}, \dots, x^{z+}, \dots, x^{w+})$$

and

$$\alpha C(x)^{-1}\alpha = (x^{a-}, x^{w-}, \dots, x^{z-}, \dots, x^{c-}, \dots, x^{v-}, \dots, x^{y-}, \dots, x^{b-}, x^{u-}, \dots).$$

It is also clear that $M_2^{\xi^*} = M_2$. Therefore, ξ^* is an automorphism of the map M_2 .

Now we consider the case of cyclic order-reversing automorphisms of a complete map. According to Lemma 2.8, we know that an element $\xi\alpha$, where, $\xi \in S_n$, is an cyclic order-reversing automorphism of a complete map only if,

$$\xi \in \mathcal{E}_{[k \frac{n_1}{k}, (2k) \frac{n-n_1}{2k}]}.$$

Our discussion is divided into two parts.

Case 3 $n_1 = n$

Without loss of generality, we can assume the cycle decomposition of ξ has the following form in this case.

$$\xi = (1, 2, \dots, k)(k+1, k+2, \dots, 2k) \cdots (n-k+1, n-k+2, \dots, n).$$

Subcase 3.1 $k \equiv 1(mod 2)$ and $k > 1$

According to Lemma 2.8, we know that $\xi^*\alpha$ is not an automorphism of maps since $o(\xi^*) = k \equiv 1(mod 2)$.

Subcase 3.2 $k \equiv 0(mod 2)$

Construct a non-orientable map $M_3 = (\mathcal{X}_{\alpha, \beta}^3, \mathcal{P}_3)$, where $X^3 = E(K_n)$ and

$$\mathcal{P}_3 = \prod_{i \in \{1, 2, \dots, n\}} (C(i))(\alpha C(i)^{-1}\alpha),$$

where, if $i \equiv 1(mod 2)$, then

$$C(i) = (i^{1+}, i^{k+1+}, \dots, i^{n-k+1+}, i^{2+}, \dots, i^{n-k+2+}, \dots, i^{i^*}, \dots, i^{k+}, i^{2k+}, \dots, i^{n+}),$$

$$\alpha C(i)^{-1}\alpha = (i^{1-}, i^{n-}, \dots, i^{2k-}, i^{k-}, \dots, i^{k+1-})$$

and if $i \equiv 0(mod 2)$, then

$$C(i) = (i^{1-}, i^{k+1-}, \dots, i^{n-k+1-}, i^{2-}, \dots, i^{n-k+2-}, \dots, i^{i^*}, \dots, i^{k-}, i^{2k-}, \dots, i^{n-}),$$

$$\alpha C(i)^{-1} \alpha = (i^{1+}, i^{n+}, \dots, i^{2k+}, i^{k+}, \dots, i^{k+1+}).$$

Where, i^{i*} denotes the empty position, for example, $(2^1, 2^{2*}, 2^3, 2^4, 2^5) = (2^1, 2^3, 2^4, 2^5)$. It is clear that $\mathcal{P}_3^{\xi\alpha} = \mathcal{P}_3$, that is, $\xi\alpha$ is an automorphism of map M_3 .

Case 4 $n_1 \neq n$

Without loss of generality, we can assume that

$$\begin{aligned} \xi &= (1, 2, \dots, k)(k+1, k+2, \dots, n_1) \cdots (n_1 - k + 1, n_1 - k + 2, \dots, n_1) \\ &\times (n_1 + 1, n_1 + 2, \dots, n_1 + 2k)(n_1 + 2k + 1, \dots, n_1 + 4k) \cdots (n - 2k + 1, \dots, n) \end{aligned}$$

Subcase 4.1 $k \equiv 0(mod 2)$

Consider the orbits of 1^{2+} and $n_1 + 2k + 1^{1+}$ under the action of $\langle \xi\alpha \rangle$, we get that

$$|orb((1^{2+})^{\langle \xi\alpha \rangle})| = k$$

and

$$|orb(((n_1 + 2k + 1)^{1+})^{\langle \xi\alpha \rangle})| = 2k.$$

Contradicts to Lemma 2.5.

Subcase 4.2 $k \equiv 1(mod 2)$

In this case, if $k \neq 1$, then $k \geq 3$. Similar to the discussion of Subcase 3.1, we know that $\xi\alpha$ is not an automorphism of complete map. Whence, $k = 1$ and

$$\xi \in \mathcal{E}_{[1^{n_1}, 2^{n_2}]}$$

Without loss of generality, assume that

$$\xi = (1)(2) \cdots (n_1)(n_1 + 1, n_1 + 2)(n_1 + 3, n_1 + 4) \cdots (n_1 + n_2 - 1, n_1 + n_2).$$

If $n_2 \geq 2$, and there exists a map $M = (\mathcal{X}_{\alpha, \beta}, \mathcal{P})$, assume the vertex v_1 in M being

$$v_1 = (1^{l_{12}+}, 1^{l_{13}+}, \dots, 1^{l_{1n}+})(1^{l_{12}-}, 1^{l_{1n}-}, \dots, 1^{l_{13}-})$$

where, $l_{1i} \in \{+2, -2, +3, -3, \dots, +n, -n\}$ and $l_{1i} \neq l_{1j}$ if $i \neq j$.

Then we get that

$$(v_1)^{\xi\alpha} = (1^{l_{12}-}, 1^{l_{13}-}, \dots, 1^{l_{1n}-})(1^{l_{12}+}, 1^{l_{1n}+}, \dots, 1^{l_{13}+}) \neq v_1.$$

Whence, $\xi\alpha$ is not an automorphism of map M , a contradiction.

Therefore, $n_2 = 1$. Similarly, we can also get that $n_1 = 2$. Whence, $\xi = (1)(2)(34)$ and $n = 4$. We construct a stable non-orientable map M_4 under the action of $\xi\alpha$ as follows.

$$M_4 = (\mathcal{X}_{\alpha,\beta}^4, \mathcal{P}_4),$$

where,

$$\begin{aligned} \mathcal{P}_4 = & (1^{2+}, 1^{3+}, 1^{4+})(2^{1+}, 2^{3+}, 2^{4+})(3^{1+}, 3^{2+}, 3^{4+})(4^{1+}, 4^{2+}, 4^{3+}) \\ & \times (1^{2-}, 1^{4-}, 1^{3-})(2^{1-}, 2^{4-}, 2^{3-})(3^{1-}, 3^{4-}, 3^{2-})(4^{1-}, 4^{3-}, 4^{2-}). \end{aligned}$$

Therefore, all cyclic order-preserving automorphisms of non-orientable complete maps are extended actions of elements in

$$\mathcal{E}_{[s, \frac{n}{s}]}, \quad \mathcal{E}_{[1, s \frac{n-1}{s}]}$$

and all cyclic order-reversing automorphisms of non-orientable complete maps are extended actions of elements in

$$\alpha\mathcal{E}_{[(2s) \frac{n}{2s}]}, \quad \alpha\mathcal{E}_{[(2s) \frac{4}{2s}]} \quad \alpha\mathcal{E}_{[1,1,2]}.$$

This completes the proof. \square

According to the Rotation Embedding Scheme for orientable embedding of a graph formalized by Edmonds in [5], each orientable complete map is just the case of eliminating the signs "+, -" in our representation for complete maps. Whence, we also get the following result for automorphisms of orientable complete maps, which is similar to Theorem 3.1.

Theorem 3.2 *All cyclic order-preserving automorphisms of orientable complete maps of order ≥ 4 are extended actions of elements in*

$$\mathcal{E}_{[s, \frac{n}{s}]}, \quad \mathcal{E}_{[1, s \frac{n-1}{s}]}$$

and all cyclic order-reversing automorphisms of orientable complete maps of order ≥ 4 are extended actions of elements in

$$\alpha\mathcal{E}_{[(2s) \frac{n}{2s}]}, \quad \alpha\mathcal{E}_{[(2s) \frac{4}{2s}]}, \quad \alpha\mathcal{E}_{[1,1,2]},$$

where, \mathcal{E}_θ denotes the conjugate class containing θ in S_n .

Proof The proof is similar to that of Theorem 3.1. For completion, we only need to construct orientable maps $M_i^O, i = 1, 2, 3, 4$ to replace these non-orientable maps $M_i, i = 1, 2, 3, 4$ in the proof of Theorem 3.1.

In fact, for cyclic order-preserving case, we only need to take M_1^O , M_2^O to be the resultant maps eliminating the signs $+$ $-$ in M_1 , M_2 constructed in the proof of Theorem 3.1.

For the cyclic order-reversing case, we take $M_3^O = (E(K_n)_{\alpha,\beta}, \mathcal{P}_3^O)$ with

$$\mathcal{P}_3 = \prod_{i \in \{1,2,\dots,n\}} (C(i)),$$

where, if $i \equiv 1(mod 2)$, then

$$C(i) = (i^1, i^{k+1}, \dots, i^{n-k+1}, i^2, \dots, i^{n-k+2}, \dots, i^{i^*}, \dots, i^k, i^{2k}, \dots, i^n),$$

and if $i \equiv 0(mod 2)$, then

$$C(i) = (i^1, i^{k+1}, \dots, i^{n-k+1}, i^2, \dots, i^{n-k+2}, \dots, i^{i^*}, \dots, i^k, i^{2k}, \dots, i^n)^{-1},$$

where i^{i^*} denotes the empty position and $M_4^O = (E(K_4)_{\alpha,\beta}, \mathcal{P}_4)$ with

$$\mathcal{P}_4 = (1^2, 1^3, 1^4)(2^1, 2^3, 2^4)(3^1, 3^4, 3^2)(4^1, 4^2, 4^3).$$

It can be shown that $(M_i^O)^{g^*} = M_i^O$, $i = 1, 2$ and $(M_i^O)^{\xi^\alpha} = M_i^O$ for $i = 3, 4$. \natural

All results in this section are useful for the enumeration of complete maps in the next section.

4. The Enumeration of complete maps on surfaces

We first consider the permutation and its stabilizer . The permutation with the following form $(x_1, x_2, \dots, x_n)(\alpha x_n, \alpha x_2, \dots, \alpha x_1)$ is called a pair permutation. The following result is obvious.

Lemma 4.1 *Let g be a permutation on the set $\Omega = \{x_1, x_2, \dots, x_n\}$ such that $g\alpha = \alpha g$. If*

$$g(x_1, x_2, \dots, x_n)(\alpha x_n, \alpha x_{n-1}, \dots, \alpha x_1)g^{-1} = (x_1, x_2, \dots, x_n)(\alpha x_n, \alpha x_{n-1}, \dots, \alpha x_1),$$

then

$$g = (x_1, x_2, \dots, x_n)^k$$

and if

$$g\alpha(x_1, x_2, \dots, x_n)(\alpha x_n, \alpha x_{n-1}, \dots, \alpha x_1)(g\alpha)^{-1} = (x_1, x_2, \dots, x_n)(\alpha x_n, \alpha x_{n-1}, \dots, \alpha x_1),$$

then

$$g\alpha = (\alpha x_n, \alpha x_{n-1}, \dots, \alpha x_1)^k$$

for some integer $k, 1 \leq k \leq n$.

Lemma 4.2 *For each permutation $g, g \in \mathcal{E}_{[k \frac{n}{k}]}$ satisfying $g\alpha = \alpha g$ on the set $\Omega = \{x_1, x_2, \dots, x_n\}$, the number of stable pair permutations in Ω under the action of g or $g\alpha$ is*

$$\frac{2\phi(k)(n-1)!}{|\mathcal{E}_{[k \frac{n}{k}]}|},$$

where $\phi(k)$ denotes the Euler function.

Proof Denote the number of stable pair permutations under the action of g or $g\alpha$ by $n(g)$ and \mathcal{C} the set of pair permutations. Define the set $A = \{(g, C) | g \in \mathcal{E}_{[k \frac{n}{k}]}, C \in \mathcal{C} \text{ and } C^g = C \text{ or } C^{g\alpha} = C\}$. Clearly, for $\forall g_1, g_2 \in \mathcal{E}_{[k \frac{n}{k}]}$, we have $n(g_1) = n(g_2)$. Whence, we get that

$$|A| = |\mathcal{E}_{[k \frac{n}{k}]}|n(g). \quad (4.1)$$

On the other hand, by Lemma 4.1, for any pair permutation $C = (x_1, x_2, \dots, x_n)(\alpha x_n, \alpha x_{n-1}, \dots, \alpha x_1)$, since C is stable under the action of g , there must be $g = (x_1, x_2, \dots, x_n)^l$ or $g\alpha = (\alpha x_n, \alpha x_{n-1}, \dots, \alpha x_1)^l$, where $l = s\frac{n}{k}, 1 \leq s \leq k$ and $(s, k) = 1$. Therefore, there are $2\phi(k)$ permutations in $\mathcal{E}_{[k \frac{n}{k}]}$ acting on it stable. Whence, we also have

$$|A| = 2\phi(k)|\mathcal{C}|. \quad (4.2)$$

Combining (4.1) with (4.2), we get that

$$n(g) = \frac{2\phi(k)|\mathcal{C}|}{|\mathcal{E}_{[k \frac{n}{k}]}|} = \frac{2\phi(k)(n-1)!}{|\mathcal{E}_{[k \frac{n}{k}]}|}. \quad \spadesuit$$

Now we can enumerate the unrooted complete maps on surfaces.

Theorem 4.1 *The number $n^L(K_n)$ of complete maps of order $n \geq 5$ on surfaces is*

$$n^L(K_n) = \frac{1}{2} \left(\sum_{k|n} + \sum_{k|n, k \equiv 0 \pmod{2}} \right) \frac{2^{\alpha(n,k)}(n-2)!^{\frac{n}{k}}}{k^{\frac{n}{k}} \left(\frac{n}{k}\right)!} + \sum_{k|(n-1), k \neq 1} \frac{\phi(k)2^{\beta(n,k)}(n-2)!^{\frac{n-1}{k}}}{n-1},$$

where,

$$\alpha(n, k) = \begin{cases} \frac{n(n-3)}{2k}, & \text{if } k \equiv 1 \pmod{2}; \\ \frac{n(n-2)}{2k}, & \text{if } k \equiv 0 \pmod{2}, \end{cases}$$

and

$$\beta(n, k) = \begin{cases} \frac{(n-1)(n-2)}{2k}, & \text{if } k \equiv 1(\text{mod}2); \\ \frac{(n-1)(n-3)}{2k}, & \text{if } k \equiv 0(\text{mod}2). \end{cases}$$

and $n^L(K_4) = 11$.

Proof According to (2.3) in Corollary 2.1 and Theorem 3.1 for $n \geq 5$, we know that

$$\begin{aligned} n^L(K_n) &= \frac{1}{2|\text{Aut}K_n|} \times \left(\sum_{g_1 \in \mathcal{E}_{[k, \frac{n}{k}]}} |\Phi(g_1)| + \sum_{g_2 \in \mathcal{E}_{[(2s), \frac{n}{2s}]}} |\Phi(g_2\alpha)| \right. \\ &\quad \left. + \sum_{h \in \mathcal{E}_{[1, k, \frac{n-1}{k}]}} |\Phi(h)| \right) \\ &= \frac{1}{2n!} \times \left(\sum_{k|n} |\mathcal{E}_{[k, \frac{n}{k}]}| |\Phi(g_1)| + \sum_{l|n, l \equiv 0(\text{mod}2)} |\mathcal{E}_{[l, \frac{n}{l}]}| |\Phi(g_2\alpha)| \right. \\ &\quad \left. + \sum_{l|(n-1)} |\mathcal{E}_{[1, l, \frac{n-1}{l}]}| |\Phi(h)| \right), \end{aligned}$$

where, $g_1 \in \mathcal{E}_{[k, \frac{n}{k}]}$, $g_2 \in \mathcal{E}_{[l, \frac{n}{l}]}$ and $h \in \mathcal{E}_{[1, k, \frac{n-1}{k}]}$ are three chosen elements.

Without loss of generality, we assume that an element $g, g \in \mathcal{E}_{[k, \frac{n}{k}]}$ has the following cycle decomposition.

$$g = (1, 2, \dots, k)(k+1, k+2, \dots, 2k) \dots ((\frac{n}{k} - 1)k + 1, (\frac{n}{k} - 1)k + 2, \dots, n)$$

and

$$\mathcal{P} = \Pi_1 \times \Pi_2,$$

where,

$$\Pi_1 = (1^{i_{21}}, 1^{i_{31}}, \dots, 1^{i_{n1}})(2^{i_{12}}, 2^{i_{32}}, \dots, 2^{i_{n2}}) \dots (n^{i_{1n}}, n^{i_{2n}}, \dots, n^{i_{(n-1)n}}),$$

and

$$\Pi_2 = \alpha(\Pi_1^{-1})\alpha^{-1}$$

being a complete map which is stable under the action of g , where $s_{ij} \in \{k+, k- | k = 1, 2, \dots, n\}$.

Notice that the quadricells adjacent to the vertex "1" can make $2^{n-2}(n-2)!$ different pair permutations and for each chosen pair permutation, the pair permutations adjacent to the vertices $2, 3, \dots, k$ are uniquely determined since \mathcal{P} is stable under the action of g .

Similarly, for each given pair permutation adjacent to the vertex $k+1, 2k+1, \dots, (\frac{n}{k}-1)k+1$, the pair permutations adjacent to $k+2, k+3, \dots, 2k$ and $2k+2, 2k+3, \dots, 3k$ and, ..., and $(\frac{n}{k}-1)k+2, (\frac{n}{k}-1)k+3, \dots, n$ are also uniquely determined because \mathcal{P} is stable under the action of g .

Now for an orientable embedding M_1 of K_n , all the induced embeddings by exchanging two sides of some edges and retaining the others unchanged in M_1 are the same as M_1 by the definition of maps. Whence, the number of different stable embeddings under the action of g gotten by exchanging x and αx in M_1 for $x \in U, U \subset \mathcal{X}_\beta$, where $\mathcal{X}_\beta = \bigcup_{x \in E(K_n)} \{x, \beta x\}$, is $2^{g(\varepsilon) - \frac{n}{k}}$, where $g(\varepsilon)$ is the number of orbits of $E(K_n)$ under the action of g and we subtract $\frac{n}{k}$ because we can choose $1^{2+}, k+1^{1+}, 2k+1^{1+}, \dots, n-k+1^{1+}$ first in our enumeration.

Notice that the length of each orbit under the action of g is k for $\forall x \in E(K_n)$ if k is odd and is $\frac{k}{2}$ for $x = i^{i+\frac{k}{2}}, i = 1, k+1, \dots, n-k+1$, or k for all other edges if k is even. Therefore, we get that

$$g(\varepsilon) = \begin{cases} \frac{\varepsilon(K_n)}{k}, & \text{if } k \equiv 1(\text{mod } 2); \\ \frac{\varepsilon(K_n) - \frac{n}{2}}{k}, & \text{if } k \equiv 0(\text{mod } 2). \end{cases}$$

Whence, we have that

$$\alpha(n, k) = g(\varepsilon) - \frac{n}{k} = \begin{cases} \frac{n(n-3)}{2k}, & \text{if } k \equiv 1(\text{mod } 2); \\ \frac{n(n-2)}{2k}, & \text{if } k \equiv 0(\text{mod } 2), \end{cases}$$

and

$$|\Phi(g)| = 2^{\alpha(n, k)} (n-2)!^{\frac{n}{k}}, \quad (4.3)$$

Similarly, if $k \equiv 0(\text{mod } 2)$, we get also that

$$|\Phi(g\alpha)| = 2^{\alpha(n, k)} (n-2)!^{\frac{n}{k}} \quad (4.4)$$

for an chosen element $g, g \in \mathcal{E}_{[k\frac{n}{k}]}$.

Now for $\forall h \in \mathcal{E}_{[1, k\frac{n-1}{k}]}$, without loss of generality, we assume that $h = (1, 2, \dots, k)(k+1, k+2, \dots, 2k) \dots ((\frac{n-1}{k}-1)k+1, (\frac{n-1}{k}-1)k+2, \dots, (n-1))(n)$. Then the above statement is also true for the complete graph K_{n-1} with the vertices $1, 2, \dots, n-1$. Notice that the quadricells $n^{1+}, n^{2+}, \dots, n^{n-1+}$ can be chosen first in our enumeration and they are not belong to the graph K_{n-1} . According to Lemma 4.2, we get that

$$|\Phi(h)| = 2^{\beta(n, k)} (n-2)!^{\frac{n-1}{k}} \times \frac{2\phi(k)(n-2)!}{|\mathcal{E}_{[1, k\frac{n-1}{k}]}|}, \quad (4.5)$$

Where

$$\beta(n, k) = h(\varepsilon) = \begin{cases} \frac{\varepsilon(K_{n-1})}{k} - \frac{n-1}{k} = \frac{(n-1)(n-4)}{2k}, & \text{if } k \equiv 1(\text{mod } 2); \\ \frac{\varepsilon(K_{n-1})}{k} - \frac{n-1}{k} = \frac{(n-1)(n-3)}{2k}, & \text{if } k \equiv 0(\text{mod } 2). \end{cases}$$

Combining (4.3) – (4.5), we get that

$$\begin{aligned}
n^L(K_n) &= \frac{1}{2n!} \times \left(\sum_{k|n} |\mathcal{E}_{[k \frac{n}{k}]}| |\Phi(g_0)| + \sum_{l|n, l \equiv 0 \pmod{2}} |\mathcal{E}_{[l \frac{n}{l}]}| |\Phi(g_1 \alpha)| \right. \\
&\quad \left. + \sum_{l|(n-1)} |\mathcal{E}_{[1, l \frac{n-1}{l}]}| |\Phi(h)| \right) \\
&= \frac{1}{2n!} \times \left(\sum_{k|n} \frac{n! 2^{\alpha(n,k)} (n-2)! \frac{n}{k}}{k^{\frac{n}{k}} (\frac{n}{k})!} + \sum_{k|n, k \equiv 0 \pmod{2}} \frac{n! 2^{\alpha(n,k)} (n-2)! \frac{n}{k}}{k^{\frac{n}{k}} (\frac{n}{k})!} \right. \\
&\quad \left. + \sum_{k|(n-1), k \neq 1} \frac{n!}{k^{\frac{n-1}{k}} (\frac{n-1}{k})!} \times \frac{2\phi(k)(n-2)! 2^{\beta(n,k)} (n-2)! \frac{n-1}{k}}{\frac{(n-1)!}{k^{\frac{n-1}{k}} (\frac{n-1}{k})!}} \right) \\
&= \frac{1}{2} \left(\sum_{k|n} + \sum_{k|n, k \equiv 0 \pmod{2}} \right) \frac{2^{\alpha(n,k)} (n-2)! \frac{n}{k}}{k^{\frac{n}{k}} (\frac{n}{k})!} + \sum_{k|(n-1), k \neq 1} \frac{\phi(k) 2^{\beta(n,k)} (n-2)! \frac{n-1}{k}}{n-1}.
\end{aligned}$$

For $n = 4$, similar calculation shows that $n^L(K_4) = 11$ by consider the fixing set of permutations in $\mathcal{E}_{[s \frac{4}{s}]}, \mathcal{E}_{[1, s \frac{3}{s}]}, \mathcal{E}_{[(2s) \frac{4}{2s}]}, \alpha \mathcal{E}_{[(2s) \frac{4}{2s}]}$ and $\alpha \mathcal{E}_{[1, 1, 2]}$. \spadesuit

For orientable complete maps, we get the number $n^O(K_n)$ of orientable complete maps of order n as follows.

Theorem 4.2 *The number $n^O(K_n)$ of complete maps of order $n \geq 5$ on orientable surfaces is*

$$n^O(K_n) = \frac{1}{2} \left(\sum_{k|n} + \sum_{k|n, k \equiv 0 \pmod{2}} \right) \frac{(n-2)! \frac{n}{k}}{k^{\frac{n}{k}} (\frac{n}{k})!} + \sum_{k|(n-1), k \neq 1} \frac{\phi(k)(n-2)! \frac{n-1}{k}}{n-1}.$$

and $n(K_4) = 3$.

Proof According to the Tutte's algebraic representation of maps, a map $M = (\mathcal{X}_{\alpha, \beta}, \mathcal{P})$ is orientable if and only if for $\forall x \in \mathcal{X}_{\alpha, \beta}$, x and $\alpha\beta x$ are in a same orbit of $\mathcal{X}_{\alpha, \beta}$ under the action of the group $\Psi_I = \langle \alpha\beta, \mathcal{P} \rangle$. Now applying (2.1) in Corollary 2.1 and Theorem 3.1, similar to the proof of Theorem 4.1, we get the number $n^O(K_n)$ for $n \geq 5$ as follows

$$n^O(K_n) = \frac{1}{2} \left(\sum_{k|n} + \sum_{k|n, k \equiv 0 \pmod{2}} \right) \frac{(n-2)! \frac{n}{k}}{k^{\frac{n}{k}} (\frac{n}{k})!} + \sum_{k|(n-1), k \neq 1} \frac{\phi(k)(n-2)! \frac{n-1}{k}}{n-1}.$$

and for the complete graph K_4 , calculation shows that $n(K_4) = 3$. \spadesuit

Notice that $n^O(K_n) + n^N(K_n) = n^L(K_n)$. Therefore, we can also get the number $n^N(K_n)$ of unrooted complete maps of order n on non-orientable surfaces by Theorem 4.1 and Theorem 4.2.

Theorem 4.3 *The number $n^N(K_n)$ of unrooted complete maps of order $n, n \geq 5$ on non-orientable surfaces is*

$$n^N(K_n) = \frac{1}{2} \left(\sum_{k|n} + \sum_{k|n, k \equiv 0 \pmod{2}} \right) \frac{(2^{\alpha(n,k)} - 1)(n-2)!^{\frac{n}{k}}}{k^{\frac{n}{k}} \left(\frac{n}{k}\right)!} \\ + \sum_{k|(n-1), k \neq 1} \frac{\phi(k)(2^{\beta(n,k)} - 1)(n-2)!^{\frac{n-1}{k}}}{n-1},$$

and $n^N(K_4) = 8$. Where, $\alpha(n, k)$ and $\beta(n, k)$ are same as in Theorem 4.1.

For $n = 5$, calculation shows that $n^L(K_5) = 1080$ and $n^O(K_5) = 45$ based on Theorem 4.1 and 4.2. For $n = 4$, there are 3 unrooted orientable maps and 8 non-orientable maps shown in the Fig.2.

Fig.2

All the 11 maps of K_4 on surfaces are non-isomorphic.

Noticing that for an orientable map M , its cyclic order-preserving automorphisms are just the orientation-preserving automorphisms of map M by definition. Now consider the action of cyclic order-preserving automorphisms of complete maps, determined in Theorem 3.2 on all orientable embeddings of a complete graph of order n . Similar to the proof of Theorem 4.2, we can get the number of non-equivalent embeddings of complete graph of order n , which is same as the result of Mull et al. in [15].

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